Name: Mahyar Pirayesh Date:__________________________

Math 10/11 Enriched: Section 7.2 Intersections with Lines and Circles

1. Find the intersection between the lines and circles. Provide your answers as coordinates:

a) $(x-2)^2+(y-5)^2=10$ & $y=x+5$

$(x-2)^2+(x)^2=10$

$2x^2-4x+4=10$

$x^2-2x-3=0$

$(x+1)(x-3)=0$

$x_1=-1 \quad x_2=3$

$y_1=4 \quad y_2=8$

$(x,y)=(-1,4),(3,8)$

b) $(x+3)^2+(y-8)^2=20$ & $y=3x+7$

$(x+3)^2+(3x-1)^2=20$

$x^2+6x+9+9x^2-6x+1=20$

$10x^2+10=20$

$x^2=1$

$x_1=1 \quad x_2=-1$

$y_1=10 \quad y_1=4$

$(x,y)=(1,10),(-1,4)$

c) $x^2+4x+y^2-6y=10$ & $y=\frac{2}{3}x-1$

$(x+2)^2+(y-3)^2=23$

$x^2+4x+4+\left(\frac{2}{3}x-4\right)^2=23$

$x^2+4x+4+\frac{4}{9}x^2-\frac{16}{3}x+16=23$

$\frac{13}{9}x^2-\frac{4}{3}x-3=0$

$13x^2-12x-27=0$

$x=\frac{12\pm\sqrt{12^2+4\cdot13\cdot27}}{26}=\frac{6\pm3\sqrt{43}}{13}$

$(x,y)=\left(\frac{6+3\sqrt{43}}{13},\frac{-9+2\sqrt{43}}{13}\right),\left(\frac{6-3\sqrt{43}}{13},\frac{-9-2\sqrt{43}}{13}\right)$

d) $3x^2-12x+5y^2+20y=17$, $y=\frac{-3}{4}x+2$

$3x^2-12x+5\left(-\frac{3}{4}x+2\right)^2+20\left(-\frac{3}{4}x+2\right)=17$

$3x^2-12x+5\left(\frac{9}{16}x^2-3x+4\right)-15x+40=17$

$3x^2-12x+\frac{45}{16}x^2-15x+20-15x+40=17$

$\frac{93x^2}{16}-42x+43=0$

$93x^2-42\times16x+43\times16=0$

$x=\frac{42\cdot16\pm\sqrt{(42\cdot16)^2-4(93)(43\cdot16)}}{2\cdot93}$

$x_1=5.99$

$x_2=1.23$

$(x,y)=(5.99,-2.5),(1.23,1.1)$

2. Find the equation of the tangent line for each of the following:

a) $x^2+y^2-8x-14y+40=0$ at the point (8,4)

b) $x^2+y^2-6x+4y=13$ & $P(k,-1)$

c)

The point $P(12,9)$ lies on the circle with equation

$$(x+3)^2+(y-1)^2=289.$$

a) Find an equation of the normal to the circle at P.

b) Determine the coordinates of the point Q, where the normal to the circle at P intersects the circle again.

d)

The figure above shows a circle that passes through the points $A(-1,10)$ and $B(7,4)$.

a) Given that AB is a diameter of the circle show that an equation for this circle is given by

$$x^2+y^2-6x-14y+33=0.$$

a) $(x-4)^2+(y-7)^2=25$

slope of radius: $\frac{4-7}{8-4}=\frac{-3}{4}$

slope of tangent: $\frac{4}{3}$ eqn: $(y-4)=\frac{4}{3}(x-8)$

b) $(x-3)^2+(y+2)^2=26$

$k^2+1-6k-4=13$

$k^2-6k=16$

$k^2-6k-16=0$

$(k-8)(k+2)=0$

$k=8,-2$

slope of radius: $\frac{1}{5}$ or $\frac{1}{-5}$

slope tangent: -5 or 5

eqn: $(y+1)=-5(x-8)$ or $(y+1)=5(x+2)$

A circle has equation

$$x^2 + y^2 + ax + by = 0,$$

where a and b are constants.

The straight lines with equations

$$y = x - 4 \qquad \text{and} \qquad x + y = 2$$

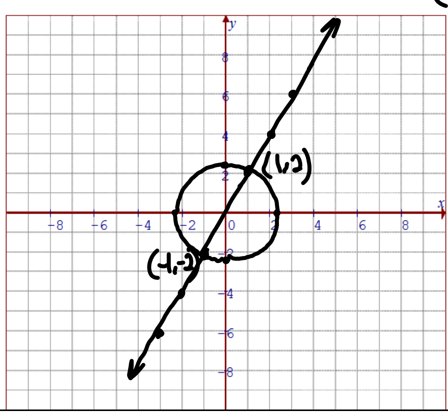
More info?

3. Determine the coordinates of intersection of each line and circle both graphically and algebraically :

a) $x^2 + y^2 = 5$ & $y = 2x$

$x^2 + 4x^2 = 5 \Rightarrow x = \pm 1 \quad y = \pm 2$

(x,y) $(1,2)$, $(-1,-2)$



b) $x^2 + y^2 = 5$ & $y = 2x - 5$

$x^2 + (2x-5)^2 = 5$

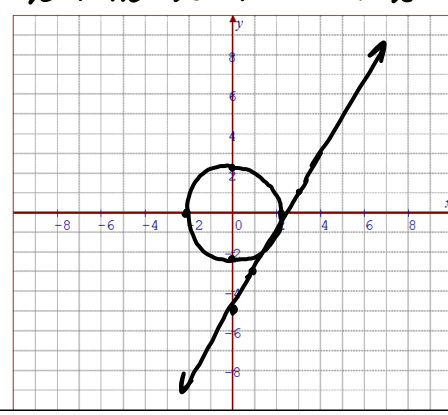
$x^2 + 4x^2 - 20x + 25 = 5 \Rightarrow x^2 - 4x + 4 = 0$

$(x-2)^2 = 0$

$x = 2$

$y = -1$

$(x,y) = (2,-1)$



c) $x^2 + y^2 = 10$ & $y = 2x - 5$

$x^2 + (2x-5)^2 = 10$

$x^2 + 4x^2 - 20x + 25 = 10$

$x^2 - 4x + 3 = 0$

$(x-1)(x-3) = 0$

$x_1 = 1, x_2 = 3$

$y_1 = -3, y_2 = 1$

$(x,y) = (1,-3), (3,1)$

d) $x^2 + y^2 = 25$ & $x - y = 8$

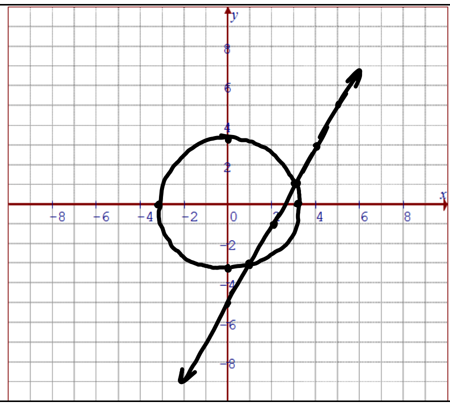
$(8+y)^2 + y^2 = 25 \qquad x = 8 + y$

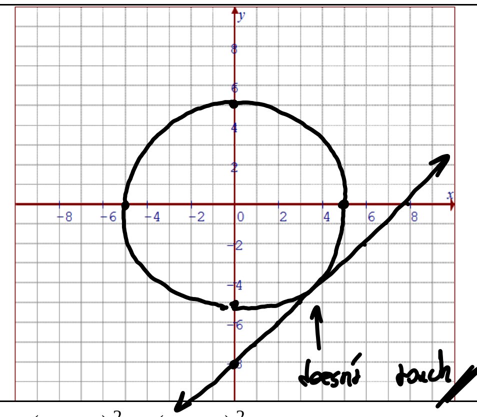
$64 + 16y + y^2 + y^2 = 25$

$2y^2 + 16y + 39 = 0$

$\emptyset$

NO SOL



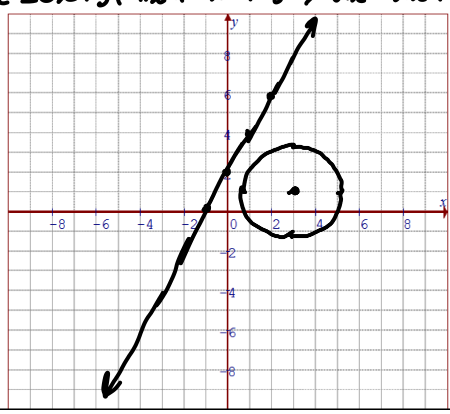


e) $(x-3)^2+(y-1)^2=5$ & $y=2x+2$

$(x-3)^2+(2x+1)^2=5$

$x^2-6x+9+4x^2+4x+1=5 \Rightarrow 5x^2-2x+5=0 \Rightarrow \Delta=\sqrt{4-100}$

$\therefore \varnothing$ NO sol



f) $(x-3)^2+(y-1)^2=5$ & $y=2x-2$

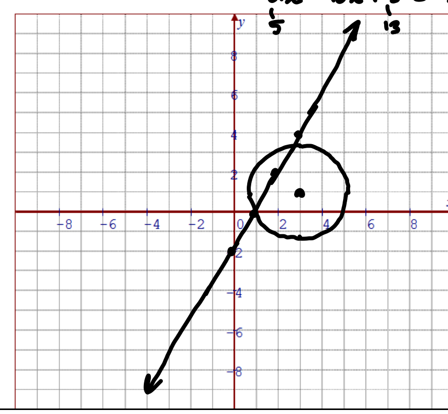
$(x-3)^2+(2x-3)^2=5 \Rightarrow x^2-6x+9+4x^2-12x+9=5$

$5x^2-18x+13=0 \Rightarrow (5x-13)(x-1)=0$

$x_1=\frac{13}{5}$ $x_2=1$

$y_1=\frac{16}{5}$ $y_2=0$

$(x,y)=\left(\frac{13}{5},\frac{16}{5}\right),(1,0)$



4. Determine all (x,y) such that $x^2+y^2=25$ and $x-y=1$

$y=x-1$

$x^2+(x-1)^2=25$

$2x^2-2x-24=0$

$x^2-x-12=0 \Rightarrow (x+3)(x-4)=0 \Rightarrow x=-3,4$ $y=-4,3$

$(x,y)=(-3,-4),(4,3)$

5. Given the circle $(x+1)^2+(y-3)^2=25$, determine the coordinates of the endpoint of a diameter with slope of $\frac{-4}{3}$. Determine also the equations of the tangents to the circle at these endpoints.

the numbers work out perfectly.

$(h,k)=(-1,3)$

$(x,y)=(2,7),(-4,-1)$

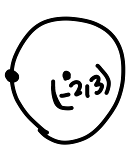
$(y-7)=\frac{3}{4}(x-2)$

$(y+1)=\frac{3}{4}(x+4)$

6. Given the equation of a circle $x^2+4x+y^2-6y=-9$, find the equation of the line tangent to the circle at point $(0,3)$

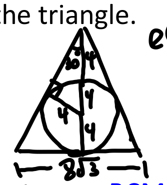
$(x+2)^2+(y-3)^2=4$

slope of radius $=\frac{3-3}{0+2}=0$

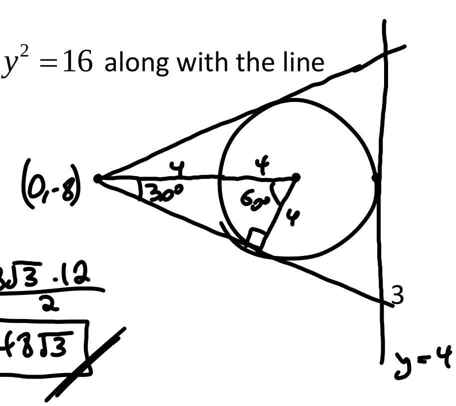


eqn: $x=0$

7. Two lines pass through the point $(0,-8)$ and are tangent to the circle $x^2+y^2=16$ along with the line $y=4$ to form a triangle. Find the area of the triangle.



equilateral triangle!



Area $=\frac{8\sqrt{3}\cdot 12}{2}$

$=48\sqrt{3}$

8. What is the intersection point between the two curves: $y=\frac{5}{x}$ and $4x^2+25y^2=100$

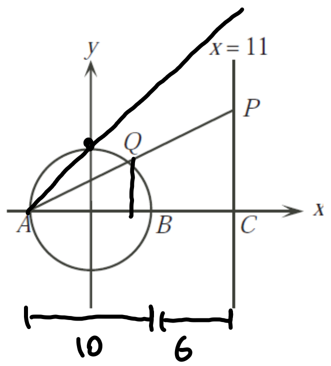
$(x,y)=\left(\frac{5\sqrt{2}}{2},\sqrt{2}\right),\left(-\frac{5\sqrt{2}}{2},-\sqrt{2}\right)$

let $x^2=A$ $\quad 4x^2+25\left(\frac{5}{x}\right)^2=100$

$4A+\frac{625}{A}=100 \Rightarrow 4A^2-100A+625=0$

$(2A-25)^2=0 \rightarrow A=\frac{25}{2} \rightarrow x^2=\frac{25}{2} \rightarrow x=\pm\frac{5}{\sqrt{2}}$, $y=\pm\sqrt{2}$

9. In the diagram, the circle $x^2+y^2=25$ intersects the x-axis at points "A" and "B". The line $x=11$ intersects the x-axis at point "C". Point "P" moves along the line $x=11$ above the x-axis and AP intersects the circle at "Q".

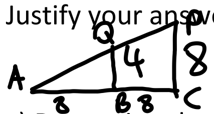


a) Determine the coordinates of "P" when $\triangle ABQ$ has maximum area. Justify your answer

Q on y-axis when max area.

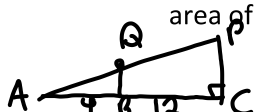
$P(11,16)$

b) Determine the coordinates of "P" when "Q" is the midpoint of AP. Justify your answer



$Q(3,y)$ $\quad y^2+3^2=25$, $y^2=16$, $y=4 \Rightarrow Q(3,4) \Rightarrow P(11,8)$

c) Determine the coordinates of "P" when the area of $\triangle AQB$ is ¼ of the area of $\triangle APC$. Justify your answer



$Q(-1,y)$ $\quad y^2+1=25$, $y=2\sqrt{6}$ $\quad P(11,4\times2\sqrt{6})=P(11,8\sqrt{6})$

10. The points $A(-8,6)$ and $B(-6,-8)$ lie on the circle $x^2+y^2=100$

a. Determine the equation of the line through "A" and "B"

$m=\frac{6+8}{-8+6}=\frac{14}{-2}=-7$ $\quad (y-6)=-7(x+8)$

$y=-7x-50$

b. Determine the equation of the perpendicular bisector of AB

$m=\frac{1}{7}$ passes through $\left(\frac{-8-6}{2},\frac{6-8}{2}\right)=(-7,-1)$

$y+1=\frac{1}{7}(x+7) \Rightarrow y=\frac{1}{7}x$

c. The perpendicular bisector of AB cuts the circle at two points, "P" in the first quadrant and "Q" in the third quadrant. Determine the coordinates of "P" and "Q"

$y=\frac{1}{7}x$ and $x^2+y^2=100$

$x^2+\left(\frac{1}{7}x\right)^2=100 \Rightarrow x^2+\frac{x^2}{49}=100 \Rightarrow 50x^2=49\cdot 100$ 2

$x=\pm7\sqrt{2}$ $\quad y=\pm\sqrt{2}$

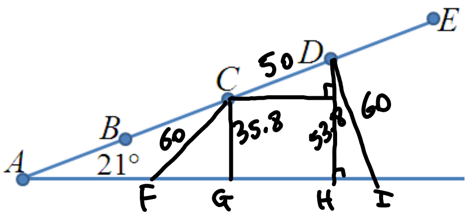
$P(7\sqrt{2},\sqrt{2})$, $Q(-7\sqrt{2},-\sqrt{2})$

d. What is the length of PQ? Justify your answer

PQ = diameter since it passes through (0,0) which is the centre of the circle.

$\therefore PQ=2\cdot10=20$

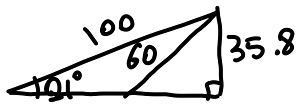
11. Streetlights A, B, C, D, and E are placed 50m apart on the main road as shown below. The light from a streetlight is effective up to a distance of 60m.



a) Determine the distance from "A" to the farthest point on the side road that is effectively illuminated

distance = radius = 50m

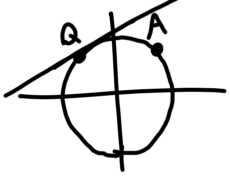
b) Determine the length of the side road that is effectively illuminated by both streetlights "C" and "D"



length = FG + GH + HI = $\sqrt{60^2-35.8^2} + 50\sin 21° + \sqrt{60^2-53.8^2}$

$= 92.7$ m

12. Points Q(-3,4) and A(3,4) lie on the circle $x^2+y^2=25$. Point P(5,10) is outside the circle. Show that line PQ intersects the circle at only one point. Also, show that $PQ^2 = PA \times PB$. P534



PQ: $y-4=\frac{3}{4}(x+3) \Rightarrow y=\frac{3}{4}x+\frac{25}{4}$

circle: $x^2+y^2=25$

$x^2+\frac{1}{16}(3x+25)^2=25$

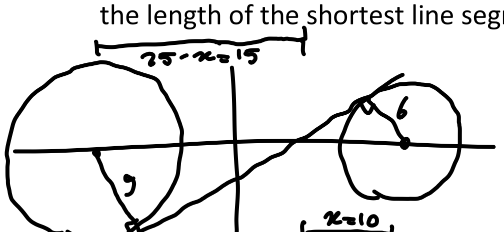
$x^2+\frac{9x^2}{16}+\frac{75x}{8}+\frac{25^2}{16}=25$

$\frac{25x^2}{16}+\frac{75x}{8}+\frac{25^2}{16}-25=0$

discriminant = $\sqrt{b^2-4ac}=0$, so only 1 solution

? No "B"!

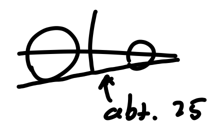
13. Let C_1 and C_2 be the circles defined by: $(x-10)^2+y^2=36$ and $(x+15)^2+y^2=81$, respectively. What is the length of the shortest line segment PQ that is tangent to C_1 at "P" and C_2 at "Q"?



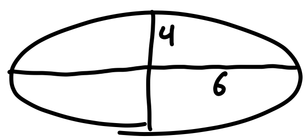
$\frac{x}{25-x}=\frac{6}{9} \Rightarrow x=10$

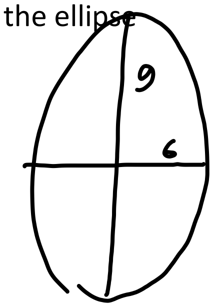
$\sqrt{10^2-6^2}+\sqrt{15^2-9^2}=20$

other possibility is longer



14. Let b_1 be the perimeter of the ellipse $\frac{x^2}{36}+\frac{y^2}{16}=1$, and let b_2 be the perimeter of the ellipse $\frac{x^2}{36}+\frac{y^2}{81}=1$. What is the ratio of b_1 to b_2





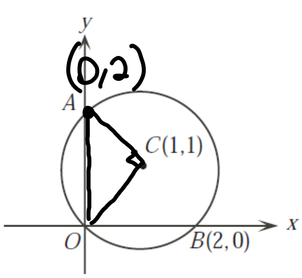
ratio of sides = $\frac{4}{6}=\frac{2}{3}$

$\frac{b_1}{b_2}=\frac{2}{3}$

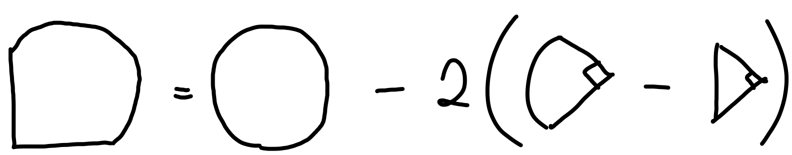
4 : 6 = 6 : 9

(Perimeter approximation: $\pi(a+b)$ also works)

15. In the diagram, the circle with center C(1,1) passes through the point (0,0), intersects the y-axis at "A" and intersects the x-axis at B(2,0). Determine the coordinates of "A" and the area of the part of the circle that lies in the first quadrant.



$r=\sqrt{2}$



$= 2\pi - 2\left(\frac{1}{2}\pi - 1\right)$

$= \pi + 2$

16. The circle $(x-p)^2+y^2=r^2$ has center "C" and the circle $x^2+(y-p)^2=r^2$ has center "D". The circles intersect at two distinct points "A" and "B", with x-coordinates "a" and "b", respectively.

a. Prove that $a+b=p$ and $a^2+b^2=r^2$

b. If "r" is fixed and "p" is then found to maximize the area of quadrilateral CADB, prove that either "A" or "B" is the origin

c. If "p" and "r" are integer, determine the minimum possible distance between "A" and "B". Find positive integers "p" and "r", each larger than 1, that gave this distance.

17. An equilateral triangle is inscribed in the ellipse whose equation is $x^2+4y^2=4$. One vertex of the triangle is (0,1), one altitude is contained in the y-axis, and the length of each side is $\sqrt{\frac{m}{n}}$, where "m" and "n" are relatively prime positive integers. Find $m+n$. AIME 2001

5. An equilateral triangle is inscribed in the ellipse whose equationis $x^2 + 4y^2 = 4$. One vertex of the triangle is $(0, 1)$, one altitude is contained in the y-axis, and the length of each side is $\sqrt{\frac{m}{n}}$, where m and n are relatively prime positive integers. Find $m + n$.

22. What is the shortest distance between two circles, the first having centre $A(5, 3)$ and radius 12, and the other with centre $B(2, -1)$ and radius 6?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Euclid 2005

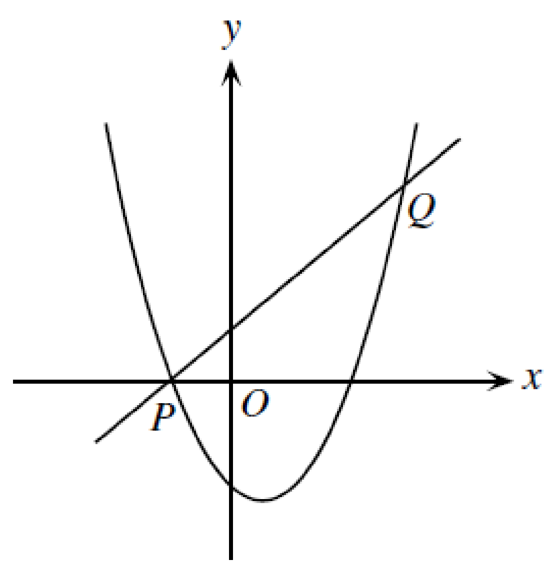
9. The circle $(x-p)^2+y^2=r^2$ has centre C and the circle $x^2+(y-p)^2=r^2$ has centre D. The circles intersect at two *distinct* points A and B, with x-coordinates a and b, respectively.

(a) Prove that $a+b=p$ and $a^2+b^2=r^2$.

(b) If r is fixed and p is then found to maximize the area of quadrilateral $CADB$, prove that either A or B is the origin.

(c) If p and r are integers, determine the minimum possible distance between A and B. Find positive integers p and r, each larger than 1, that give this distance.

In the diagram, the line $y=x+1$ intersects the parabola $y=x^2-3x-4$ at the points P and Q. Determine the coordinates of P and Q.



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18. Let C_1 and C_2 be circles defined by

$$(x-10)^2+y^2=36$$

and

$$(x+15)^2+y^2=81,$$

respectively. What is the length of the shortest line segment $\overline{PQ}$ that is tangent to C_1 at P and to C_2 at Q?

(A) 15 **(B)** 18 **(C)** 20 **(D)** 21 **(E)** 24

1.

Problem 4. Find a point (u, v) on the ellipse with equation $x^2+2y^2=1$ such that u and v are rational, and each, when expressed as a reduced fraction, has a denominator greater than 1000. Hint: Consider the line with slope m that passes through the point $(-1, 0)$.

Substitute $(x+1)m$ for y in $x^2+2y^2=1$. After a little simplification, we obtain

$$(1+2m^2)x^2+4m^2x+2m^2-1=0.$$

Now we could use the Quadratic Formula, or even, unusually, factorization, to solve for x. But this is not necessary. For the product of the roots is $(2m^2-1)/(1+2m^2)$, and -1 is one of the roots, so the other root is given by $u=(1-2m^2)/(1+2m^2)$. The corresponding v is given by $v=2m/(1+2m^2)$.

Note that if m is rational, then u and v are rational. (Parenthetically, if (u,v) lies on the ellipse, with u and v rational, with $u \neq -1$, then the slope of the line joining $(-1,0)$ to (u,v) is rational. So all rational points (u,v) on the ellipse except for $(-1,0)$ can be obtained through this process with m rational.)

Now everything is easy. Take for example $m=100$. That gives $u=19999/20001$ and $v=200/20001$.

Comment. More interestingly, the same process can be used with the circle $x^2+y^2=1$. We find that apart from $(-1,0)$, all the rational points (u,v) on the unit circle are given by $u=(1-m^2)/(1+m^2)$, $v=2m/(1+m^2)$, where m ranges over the rationals.

18. Let C_1 and C_2 be circles defined by

$$(x-10)^2+y^2=36$$

and

$$(x+15)^2+y^2=81,$$

respectively. What is the length of the shortest line segment $\overline{PQ}$ that is tangent to C_1 at P and to C_2 at Q?

(A) 15 **(B)** 18 **(C)** 20 **(D)** 21 **(E)** 24

1. The points $A(-8,6)$ and $B(-6,-8)$ lie on the circle $x^2+y^2=100$.

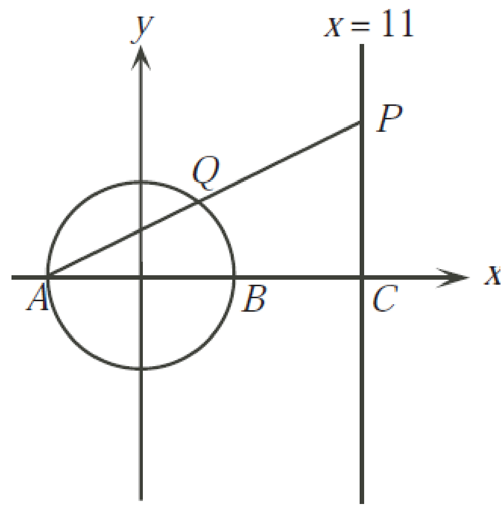
   (a) Determine the equation of the line through A and B.

   (b) Determine the equation of the perpendicular bisector of AB.

   (c) The perpendicular bisector of AB cuts the circle at two points, P in the first quadrant and Q in the third quadrant. Determine the coordinates of P and Q.

   (d) What is the length of PQ? Justify your answer.

2. In the diagram, the circle $x^2 + y^2 = 25$ intersects the x-axis at points A and B. The line $x = 11$ intersects the x-axis at point C. Point P moves along the line $x = 11$ above the x-axis and AP intersects the circle at Q.



(a) Determine the coordinates of P when $\triangle AQB$ has maximum area. Justify your answer.

(b) Determine the coordinates of P when Q is the midpoint of AP. Justify your answer.

(c) Determine the coordinates of P when the area of $\triangle AQB$ is $\frac{1}{4}$ of the area of $\triangle APC$. Justify your answer.

(c) Determine all (x, y) such that $x^2 + y^2 = 25$ and $x - y = 1$.

4. Which of the following is the centre of the conic section whose equation is:

$$4x^2 - 9y^2 + 16x + 90y - 245 = 0$$

*a) $(-2, 5)$ b) $(2, -5)$ c) $(2, 5)$
d) $(5, -2)$ e) $(5, 2)$

1. ______________________________

6. Given the equation of a circle, $x^2 + 4x + y^2 - 6y = -9$, find the equation of a line tangent to the circle at the point $(0, 3)$.
$x = 0$

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14. The two lines which pass through the point $(0, -8)$ and are tangent to the circle $x^2 + y^2 = 16$ along with the line $y = 4$ form a triangle. Find the area of the triangle. $48\sqrt{3}$

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4. For the curves $y = \dfrac{5}{x}$ and $4x^2 + 25y^2 = 100$ the y-coordinate of the point of intersection in the first quadrant is:

∗a) $\sqrt{2}$ b) 2.5 c) $\frac{5}{3}$ d) $\frac{5}{4}$ e) 5

Try to solve the following problem using concepts of coordinate geometry: In the diagram, "C" is the center of the circle. AD is tangent to the circle at point "D".

In the diagram C is the centre of the circle, AD is tangent to the circle at D, $AD = 2$, $\overleftrightarrow{AC}$ is a straight line, and $AB = \frac{3}{2}$. The length of BC is:

a) $\frac{1}{2}$ b) $\frac{1}{4}$ c) $\frac{3}{8}$

∗d) $\frac{7}{12}$ e) $\frac{7}{2}$